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Let G be a finite undirected graph with no loops or multiple edges. We define the Laplacian matrix of G, $\Delta(G)$, by Δ_{ii} = degree of vertix i and Δ_{ij} = -1 if there is an edge between vertex i and vertex j. In this paper we relate the structure of the graph G to the eigenvalues of $\Delta(G)$; in particular we prove that all the eigenvalues of $\Delta(G)$ are nonnegative, less than or equal to the number of vertices, and less than or equal to twice the maximum vertex degree. Precise conditions for equality are given.

1. Introduction

Let G be a finite undirected graph with no loops or multiple edges. We define the <u>Laplacian</u> matrix of G, $\Delta(G)$, by Δ_{ii} = the degree of vertex i and Δ_{ij} = -1 if there is an edge between vertex i and vertex j. This matrix is discussed by Harary [5]. Our name for Δ is chosen because Δ arises in numerical analysis as a discrete analog of the Laplacian operator [3]. In this paper we relate the structure of the graph G to the eigenvalues of $\Delta(G)$; in particular, we prove that all the eigenvalues of Δ are non-negative, less than or equal to the number of vertices, and less than or equal to twice the maximum vertex degree.

There is a considerable body of literature relating the eigenvalues of the adjacency matrix of a graph to its structure [6]; except for Fisher's paper [2], little seems to be known about the Laplacian.

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2. Preliminaries

Our basic graph theory reference is Harary [5]. The definitions of Δ and E, as well as Lemma 1, are taken from Chapter 13 of Harary. To define E, the <u>vertex-edge incidence matrix</u>, we first orient G. Then $E_{ij} = 1$ if edge j points toward vertex i, $E_{ij} = -1$ if edge j points away from vertex i, and $E_{ij} = 0$ otherwise. Let E^* denote the transpose of E.

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Lemma 1. $\triangle = EE^*$.

<u>Proof.</u> Two distinct rows of E will have a non-zero entry in the same column if and only if an edge joins the corresponding vertices; the corresponding entry will be 1 in one row and -1 in the other, giving a product of -1.

We will also need to consider the matrix N defined by $N = E^*E$. The important property of N is that if λ is a non-zero eigenvalue of Δ , then it is also an eigenvalue of N, and conversely. In fact, if $\Delta x = \lambda x$ with $\lambda \neq 0$, then $NE^*x = E^*\Delta x = \lambda E^*x$ so that λ is an eigenvalue of N with eigenvector E^*x . The matrix N of course depends on the choice of orientation; we will vary the orientation as needed. In particular, if G is a bipartite graph, we may point all edges toward vertices of one class. Then all entries of N are non-negative; in fact N = 2I + A, where A is the adjacency matrix of the line graph of G. Results about line graphs of bipartite graphs thus translate directly into the present context [1].

If M is a matrix, let $\rho(M)$ denote the spectral radius of M.

Let \overline{G} denote the graph complementary to G. That is, \overline{G} has the same set of vertices as G, and vertices v and w are joined in \overline{G} if and only if they are not joined in G.

Let K_n denote the complete graph on n vertices.

3. The Global Structure of G

In this section we obtain bounds for the eigenvalues of $\Delta(G)$ in terms of the number of vertices and the number of components of G.

Lemma 2. The eigenvalues of $\Delta(K_n)$ are 0, with multiplicity 1, and n, with multiplicity n-1.

<u>Proof:</u> Let u be the vector with all components equal to 1; then $\Delta(K_n)u = 0.$ If x is any vector orthogonal to u, it may be easily verified that $\Delta(K_n)x = nx.$ Q.E.D.

Theorem 1. If the graph G has n vertices, and λ is an eigenvalue of $\Delta(G)$ then $0 \le \lambda \le n$. The multiplicity of 0 equals the number of components of G; the multiplicity of n is equal to one less than the number of components of \overline{G} .

<u>Proof.</u> Suppose λ is an eigenvalue of Δ . Then for some vector \mathbf{x} , with $\|\mathbf{x}\| = 1$, $\Delta \mathbf{x} = \lambda \mathbf{x}$. Thus $\lambda = (\lambda \mathbf{x}, \mathbf{x}) = (\Delta \mathbf{x}, \mathbf{x}) = (\mathbf{E}\mathbf{E}^*\mathbf{x}, \mathbf{x}) = \|\mathbf{E}^*\mathbf{x}\|^2$. Therefore λ is real and non-negative.

Let the vertices $v_1, ..., v_K$ be the vertices of a connected component of G; then the sum of the corresponding rows of E is O, and any K-l of these rows are independent. Therefore the nullity of E, and thus of EE^* , is equal to the number of components of G.

If G has n vertices, then $\Delta(G) + \Delta(\overline{G}) = \Delta(K_n)$. If u is the vector with all components 1, then $\Delta(G)u = \Delta(\overline{G})u = \Delta(K_n)u = 0$. If $\Delta(G)x = \lambda x$ for some vector x orthogonal to u, then using Lemma 2 we have $\Delta(\overline{G})x = \Delta(K_n)x - \Delta(G)x = (n-\lambda)x$. Since the eigenvalues of $\Delta(\overline{G})$ are also non-negative, we must have $\lambda \leq n$. Moreover $\lambda = n$ if and only

if $\Delta(\overline{G})x = 0$, and the dimension of the space of such vectors is one less than the nullity of $\Delta(\overline{G})$ (since all such x are orthogonal to u). Q.E.D.

Corollary. If G has n vertices, and $\lambda = n$ is an eigenvalue of $\Delta(G)$, then G is connected.

<u>Proof.</u> If G were not connected, then G would be, and by the theorem n could not be an eigenvalue of $\Delta(G)$.

Q.E.D.

4. The Local Structure of G

In this section we obtain an upper bound for the eigenvalues of $\Delta(G)$ in terms of vertex degrees.

Before proceeding we need to recall a few facts from the theory of non-negative matrices; our basic reference is Chapter XIII of Gantmacher [4]. Briefly, a matrix M is said to be non-negative if $M_{ij} \geq 0$ for all i and j. If M is a matrix, denote by M^{\dagger} the matrix obtained by replacing each entry of M by its absolute value. If M is irreducible, and λ is an eigenvalue of M, then $|\lambda| \leq \rho(M^{\dagger})$, with equality if and only if $M = e^{i\phi}DM^{\dagger}D^{-1}$ where $D^{\dagger} = I$. For an irreducible non-negative matrix M, $\rho(M) \leq$ the maximum row sum with equality if and only if all row sums are equal.

Theorem 2. Let G be a graph. Then $\rho(\triangle(G)) \leq \text{Max} (\text{deg } v + \text{deg } w)$ where the maximum is taken over all pairs of vertices (v,w) joined by an edge of G. If G is connected, then equality holds if and only if G is bipartite and the degree is constant on each class of vertices.

Proof. We will work with the matrix N rather than Δ .

First consider a connected graph G, then N is irreducible, and thus $\rho(N) \leq \rho(N^{4}) \leq maximum \ row \ sum \ of \ N^{4}$. But if e is an edge of G joining vertices v and w, then the row sum in the row corresponding to e is deg v * deg w. The inequality is thus established for connected graphs.

If G is bipartite, then we may orient G with all edges pointing toward the vertices in one of the two classes; thus $N(G) = N^{+}(G)$. Then $\rho(N) = \max$ row sum if and only if all row sums are equal; i.e., if and only if the condition of the theorem holds. Equivalently, equality holds if and only if the line graph of G is regular.

If G is not bipartite, then we will show that $P(N) < P(N^{\dagger})$, so that equality cannot hold in the theorem. In fact, suppose $N = e^{i\phi}DN^{\dagger}D^{-1}$. Then since $N_{ii} = 2$, we have $2 = e^{i\phi}\cdot D_{ii}\cdot 2\cdot D_{ii}^{-1}$, so that $e^{i\phi} = 1$. Now suppose that the edges $1, \ldots, K$ form an odd cycle (if no odd cycle exists, then G is bipartite); we may orient G so that the corresponding entries of N are -1. Then $N_{12} = -1 = D_{11} \cdot 1 \cdot D_{22}^{-1}$, so that $D_{22} = -D_{11}$; continuing around the cycle we have $D_{11} = -D_{11}$, contradicting the requirement that $D^{\dagger} = I$. Therefore, if G is not bipartite, equality cannot hold in the theorem.

If G is not connected, the inequality, and the corresponding equality statement, follow by applying the theorem to each component separately.

Q.E.D.

Corollary. Let G be a connected graph. Then $\rho(\Delta(G)) \leq$ twice the maximum vertex degree with equality if and only if G is a regular bipartite graph.

Proof. This is a special case of the theorem

Q.E.D.

5. Explicit Computations

Theorems 1 and 2 were conjectured from explicit computations with eigenvalues; many of these were done on a digital computer. Some of these results are stated below; the reader may verify them without difficulty.

If G is the complete bipartite graph $K_{m,n}$, then the eigenvalues of $\Delta(G)$ are m*n, m, n, 0 with respective multiplicities 1, n-1, m-1, 1.

If G is the cycle with n vertices, then the eigenvalues of $\Delta(G)$ are $4\sin^2\frac{\pi K}{n}$, $K=1,2,\ldots,n$.

If G is the path with n vertices, the eigenvalues of $\Delta(G)$ are $4\sin^2\frac{\pi K}{2n}$, $K=0,1,\ldots,n-1$.

If G is the wheel with n+1 vertices, the eigenvalues of $\Delta(G)$ are n+1, 1, and 1 + $4\sin^2\frac{\pi K}{n}$, K = 1,2,...,n-1.

References

- [1] M. Doob, On characterizing certain graphs with four eigenvalues by their spectra, Linear Algebra and Appl. 3 (1970), 461-482.
- [2] M. E. Fisher, On hearing the shape of a drum, J. Comb. Theory 1 (1966), 105-125.
- [3] G. I. Forsythe and W. R. Wasow, Finite-difference methods for partial differential equations, Wiley, New York, 1960.
- [4] F. R. Gantmacher, The theory of matrices, Volume II, Chelsea, New York, 1959.
- [5] F. Harary, Graph theory, Addison-Wesley, New York, 1969.
- [6] A. J. Hoffman, Some recent results on spectral properties of graphs, in Beitrage zur Graphentheorie, Teubner, Leipzig, 1968.

Footnotes

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